

Chapter 5 *Modern Portfolio Theory*

Introduction

One of the major concepts that most investors should be aware of is the relationship between the risk and the return of a financial asset. It is common knowledge that there is a positive relationship between the risk and the expected return of a financial asset. In other words, when the risk of an asset increases, so does its expected return. What this means is that if an investor is taking on more risk, he/she is expected to be compensated for doing so with a higher return. Similarly, if the investor wants to boost the expected return of the investment, he/she needs to be prepared to take on more risk.

In this chapter, we will use the concept of Modern Portfolio Theory to derive the relationship between the risk and return of a portfolio of financial assets. We will look at three different types of portfolios that will form the foundation in helping us understand the concept and logic behind the Capital Asset Pricing Model (CAPM). The CAPM is one of the most commonly used tools in the securities industry in pricing a financial security.

We will derive the relationship between the risk and return of a portfolio (i.e. risk and return tradeoff) by looking at a series of three portfolios. We will begin by looking at the risk and return relationship of specific portfolios, and then generalize based on these findings. The three portfolios that we will examine in this chapter are:

1. A portfolio containing a risky asset and a risk-free asset.
2. A portfolio containing two risky assets.
3. A portfolio containing two risky assets and a risk-free asset.

Before we proceed with the discussions of the risk and return relationship of these three portfolios, we need to first clarify the notion of risky and risk-free assets.

Risky vs. Risk-free Assets

The classifications of risky and risk-free assets are based on relative terms and not on absolute terms. It is important to note that no financial asset can be completely risk-free.

A risk-free asset is defined as an asset that has the lowest level of risk among all the available assets. In other words, it is “risk-free” relative to the other assets.

A common practice among investment analysts is to pick a U.S. Treasury security as a representation of a risk-free asset. This is because there is virtually no chance that the U.S. federal government will go “bankrupt” since it has taxing power to raise money to pay the interest payments and redeem the securities when due. However, there is no consensus on which U.S. Treasury security should be used. The two most popular choices are the 3-month T-bill and the 30-year T-bond.

The reason the 3-month T-bill is a favorite among some investment analysts is because it is the shortest Treasury security available. As a result, it is not sensitive to changes in interest rate and inflation. However, other analysts favor the 30-year T-bond because it reflects the investment horizon of most investors even though it is more sensitive to interest rate and inflation changes.

The rate of return of a risk-free asset is generally known as the risk-free rate. This is an important return for most investors because it is often used as a benchmark to measure the return of the other financial assets. The risk-free rate represents the lowest level of return an investor expects to receive. And we know an investor will ask for a higher return as he/she takes on more risk (relative to a risk-free asset). We can rewrite the (expected) return of a risky asset as follows:

$$\begin{aligned} E(R_r) &= R_{rf} + [E(R_r) - R_{rf}] \\ &= \textit{minimum compensation} + \\ &\quad \textit{compensation for taking additional risk (i.e. risk premium)} \end{aligned}$$

In the above formula, we know the first component (the risk-free rate) represents the minimum compensation an investor can expect to receive (or it represents a guaranteed return). The second component measures the difference between the expected return of the risky asset and a risk-free asset, which represents the additional compensation the investor needs in order to take on those additional risks. This difference between the

expected return of a risky security and a risk-free security is known as the *risk premium* of the risky security.

The concept of breaking the return of a risky asset into the two components (i.e. minimum compensation and risk compensation) will become prominent in our discussions of the risk and return relationship of a portfolio.

A Portfolio Containing a Risky Asset and a Risk-free Asset

We will begin by looking at a portfolio in the simplest investment environment where there are only two types of asset available: a risky asset and a risk-free asset. In other words, when an investor decides to put together a portfolio, those are the only two types of assets available.

Suppose an investor is putting together a portfolio that contains both types of assets: w proportion of the portfolio is made up of the risky asset and $(1-w)$ is made up of risk-free asset. As a result, the return of the portfolio (R_p) is defined as follows:

$$R_p = w \cdot R_r + (1-w) \cdot R_{rf}$$

Where R_r = return of the risky asset

R_{rf} = return of the risk-free asset, i.e. the risk-free rate

We can determine the expected return and the standard deviation (i.e. risk) of the investor's portfolio as follows:

$$\begin{aligned} E(R_p) &= w \cdot E(R_r) + (1-w) \cdot R_{rf} \\ &= R_{rf} + w \cdot [E(R_r) - R_{rf}] \end{aligned}$$

$$\sigma_p = \sqrt{w^2 \cdot \sigma_r^2 + (1-w)^2 \cdot \sigma_{rf}^2} = w \cdot \sigma_r$$

As we have discussed earlier, there is a relationship between risk and return of a portfolio. In order to look at that relationship, we need to somehow combine the two

equations expressing the expected return and standard deviation of a portfolio that we have derived above.

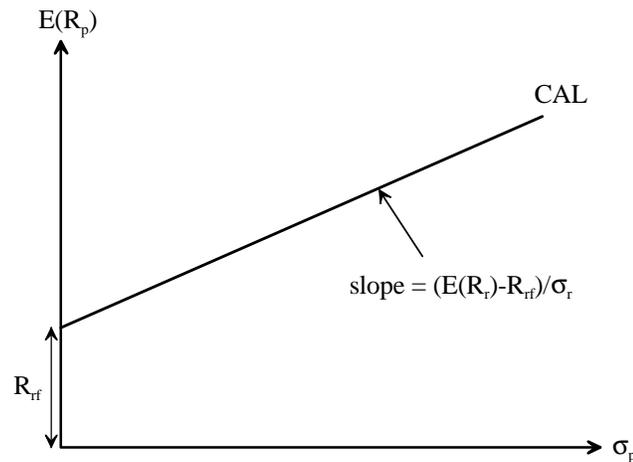
First, we need to express the weight (or proportion), w , in terms of the standard deviation of the portfolio and the risky asset:

$$w = \frac{\sigma_p}{\sigma_r}$$

We can then substitute the w as defined in the formula for the expected return of the portfolio as follows:

$$E(R_p) = R_{rf} + \frac{\sigma_p}{\sigma_r} \cdot [E(R_r) - R_{rf}] = R_{rf} + \frac{[E(R_r) - R_{rf}]}{\sigma_r} \cdot \sigma_p$$

The equation above represents the risk and return relationship of a portfolio with a risky asset and a risk-free asset. It is also known as the Capital Allocation Line (CAL), and the following is a graphical representation of the CAL:



We know the CAL represents the risk and return relationship of a portfolio. However, we are more interested in the reasoning behind the equation. As we have discussed earlier, $E(R_r) - R_{rf}$ is the risk premium of a risky asset, which represents the compensation

given to an investor for taking risk (σ_r) additional to those of a risk-free asset. As a result, the slope coefficient as presented below represents the price for risk:

$$\frac{E(R_r) - R_{rf}}{\sigma_r}$$

As a result, the total amount of compensation “received” by the investor will depend on the amount of risk taken in the portfolio (as represented by σ_p). The second component of the CAL, as presented below, represents the total compensation received for taking on the additional amount of risk:

$$\frac{E(R_r) - R_{rf}}{\sigma_r} \cdot \sigma_p$$

Once again, we can look at the relationship between risk and return of a portfolio containing a risky asset and a risk-free asset (as represented by the CAL) by breaking down the return of a portfolio into two components. From the equation of the CAL, we can tell that the expected return of a portfolio simply contains a minimum return (i.e. R_{rf}) plus the compensation for taking on additional risk.

A Portfolio Containing Multiple Risky Assets

At this point in time, we have focused on a portfolio with a risky asset and a risk-free asset. What we will do now is to look at a portfolio with multiple risky assets and no risk-free assets. In this case, the return of the portfolio (R_p) will be defined as follows:

$$R_p = w_1 R_1 + w_2 R_2 + \dots + w_n R_n = \sum w_i R_i$$

As a result, the expected return of the portfolio is simply the weighted average of the expected return of the individual assets:

$$E(R_p) = \sum w_i \times E(R_i)$$

Care has to be taken when we are dealing with the risk level of a portfolio. The formula for the standard deviation of a portfolio will get more complicated as the number of assets in the portfolio increases. To simplify the environment (and the math), we will only look at a portfolio with **two** risky assets. The standard deviation of the returns of a portfolio with two risky assets is determined as follows:

$$\sigma_p = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_1 \sigma_2 \rho}$$

Where $w_1 + w_2 = 1$

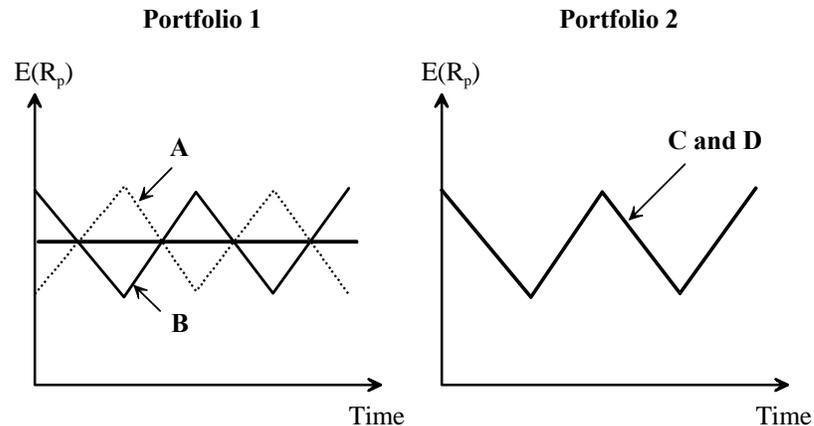
As we have discussed earlier, it is possible for an investor to reduce the risk level of a portfolio by including more and more assets in the portfolio. In other words, the investor is able to reduce the risk of the portfolio through diversification. What is the factor that drives the effectiveness of the diversification of a particular portfolio? Based on the formula for the risk of a portfolio as defined above, the effectiveness of the diversification of the portfolio depends on the correlation coefficient (ρ) of the two assets in the portfolio.

The correlation coefficient represents the direction and strength of the relationship between two assets, and it takes on a value between -1 and 1 . The signs of the correlation coefficient indicate the direction of the relationship: a positive sign indicates that there is a positive relationship between the two assets, while a negative sign indicates that there is a negative relationship between the two assets. The strength of the relationship is indicated by the absolute value of the correlation coefficient: the closer the number is to 1 the stronger the relationship, while the closer the number is to 0 the weaker the relationship. We can generalize our discussions of the correlation coefficient as follows:

- a. $\rho = 1$ means that there is an exact positive relationship between the two securities, i.e. the two move in the same direction.
- b. $\rho = 0$ means that there is no relationship between the two variables, i.e. the relationship is random.
- c. $\rho = -1$ means that there is an exact negative relationship between the two securities, i.e. the two move in opposite direction.

The relationship between correlation coefficient and the effectiveness of diversification

As we have mentioned earlier, the correlation coefficient between two financial assets plays a major role in determining the effectiveness of diversifying a portfolio. We will illustrate this concept with the following two charts that depict the relationships of two different pairs of assets in two different portfolios.



From the first chart, we can see that Portfolio 1 contains two assets, A and B. The correlation coefficient between assets A and B is -1 because they move in the exact opposite direction. In other words, when the return of one asset goes up, the return of the other asset will go down by the exact same amount; and vice versa. In this particular portfolio, the investor has invested half the money in asset A and the other half in asset B. As a result, the volatility of the two assets cancel one another completely and this leads to a situation where Portfolio 1 has no volatility at all. In this particular case, diversification has been very effective. On the other hand, Portfolio 2 contains two assets, C and D, that has a perfect positive correlation (of $+1$). In other words, the returns of the two assets move in perfect tandem. Suppose the investor also invests half the amount of money in asset C and the other half in asset D. In this case, we can see from the second chart that the return of the portfolio mimics that of the two assets. As a result, Portfolio 2 contains the same amount of risk as those of assets C and D, which means that diversification has not been effective at all.

Determining the correlation coefficient between two financial assets

Based on the two scenarios we have seen above, we know that diversification is most effective when the correlation between the assets is -1 and it is least effective when the correlation is $+1$. As a result, investors that want to reduce the risk of their portfolio as much as possible need to look for pairs of assets that have a correlation close to -1 . However, how does an investor determine the correlation coefficient between two assets? There are two ways of doing it: one way is to determine it based on the probability distribution of the future returns of the two assets, and the other way is based on historical information of realized returns of the two assets. The formulas for determining the correlation coefficient using the two techniques are as follows:

Based on probability distribution of future returns

$$\rho = \frac{\sum (x_i - E(x))(y_i - E(y))P_i}{\sigma_x \sigma_y}$$

Based on historical realized returns

$$r = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{\sqrt{(n \sum x_i^2 - (\sum x_i)^2)(n \sum y_i^2 - (\sum y_i)^2)}}$$

Since the second technique is based on a sample of historical returns of the two assets, the formula will provide the sample correlation coefficient (r) of the two assets rather than the population or “true” correlation coefficient (ρ). It might not be a true representation of the “true” correlation coefficient. As a result, care has to be taken when interpreting the meaning of a sample correlation coefficient.

Sample Correlation Coefficient	Interpretation
$0.3 \leq r \leq 1.0$	Positive relationship
$-0.3 \leq r \leq 0.3$	Random relationship
$-1.0 \leq r \leq -0.3$	Negative relationship

Despite the problems of using a sample of historical returns to estimate the correlation coefficient between two assets, it remains a very popular technique among investors and investment analysts because the formula for doing it has already been programmed in most calculators and spreadsheet programs.

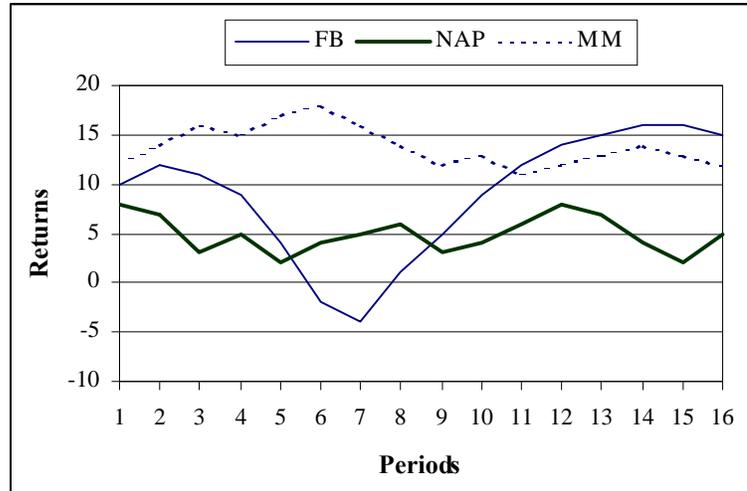
Example: Determining the correlation coefficients for three pairs of assets

You recently received the annual reports of 3 of your favorite firms: Fizzle Brewery (FB), New Age Publishing (NAP), and Mommy and Me, Inc (MM). The following are the quarterly stock returns of the three firms from 1993 to 1996:

	Period	FB	NAP	MM
1993	Quarter 1	10	8	12
	Quarter 2	12	7	14
	Quarter 3	11	3	16
	Quarter 4	9	5	15
1994	Quarter 1	4	2	17
	Quarter 2	-2	4	18
	Quarter 3	-4	5	16
	Quarter 4	1	6	14
1995	Quarter 1	5	3	12
	Quarter 2	9	4	13
	Quarter 3	12	6	11
	Quarter 4	14	8	12
1996	Quarter 1	15	7	13
	Quarter 2	16	4	14
	Quarter 3	16	2	13
	Quarter 4	15	5	12

(a) *Plot the returns of the 3 stocks over time and determine the relationship between each pair of stocks.*

We can easily plot the returns of the 3 stocks over the 4-year period from 1993 to 1996 using a spreadsheet program such as Microsoft Excel. The following chart depicts the movements of the returns of the 3 stocks in that 4-year period:



Based on the chart above, it is not easy identifying the relationships among the 3 stocks. Using our best estimation, we are concluding that there is positive relation between FB and NAP; a negative relationship between FB and MM; and a negative relationship between NAP and MM.

(b) Compute the correlation coefficient for the 3 pairs of stocks. Do they coincide with your findings in (a)?

Following the procedures outlined in Appendix 5, we can use either a scientific or a financial calculator to determine the sample correlation (r) for the 3 pairs of stocks:

$$r_{FB,NAP} = 0.1763 \quad r_{FB,MM} = -0.6123 \quad r_{NAP,MM} = -0.4406$$

The sample correlation for the 3 pairs of stocks match the “findings” in (a) in terms of the direction of the relationship.

(c) If you are going to invest in only 2 of the 3 stocks and your goal is to minimize the risk of the portfolio, which 2 stocks will you pick?

From our earlier discussion, we know that diversification is most effective when the correlation coefficient is -1 and least effective when the correlation coefficient is $+1$. In this case, you would want to pick the pair of stocks that has a correlation coefficient closest to -1 , which will be Fizzle Brewery and Mommy and Me, Inc.

Deriving the relationship between risk and return for a portfolio with 2 risky assets

In an earlier section, we showed that the formulas for the expected return and standard deviation of a portfolio with 2 risky assets are as follows:

$$E(R_p) = w_1E(R_1) + w_2E(R_2)$$

$$\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho}$$

Unlike the situation with the portfolio with a risky asset and a risk-free asset, it is not as easy to express the relationship between risk and return as a single formula for portfolios with 2 risky assets. We will use a hypothetical portfolio of 2 stocks (Turtle Computer and Fantasy Palace) to help us illustrate the risk and return relationship of such a portfolio. We can determine the relationship graphically by plotting the expected returns and standard deviations for every possible combinations of w_1 (i.e. proportion invested in Turtle Computer) and w_2 (i.e. proportion invested in Fantasy Palace). The characteristics of the 2 stocks are as follows:

	Turtle Computer	Fantasy Palace
Expected return, $E(R_i)$	15%	10%
Standard deviation, σ_i	40%	30%

Case 1: The correlation between Turtle Computer and Fantasy Palace is zero, i.e. $\rho=0$

We will look at the case where there is a random relationship between the two stocks. In other words, the correlation coefficient is zero between the two stocks. Given this piece of information, we can determine the expected return and the standard deviation of the portfolio using the two formulas given above. We will first determine the expected return and standard deviation of a portfolio with the entire fund invested in Turtle Computer (i.e. $w_1 = 1$ and $w_2 = 0$). The following are the results:

$$E(R_p) = (1 \times 0.15) + (0 \times 0.10) = 0.15$$

$$\sigma_p = \sqrt{(1^2 \times 0.4^2) + (0^2 \times 0.3^2) + (2 \times 1 \times 0 \times 0.4 \times 0.3 \times 0)} = 0.40$$

Next, we will look at the situation where 80% of the fund is invested in Turtle Computer and 20% of it invested in Fantasy Palace (i.e. $w_1 = 0.8$ and $w_2 = 0.2$). Using the same information, we can determine the expected return and the standard deviation of the portfolio as follows:

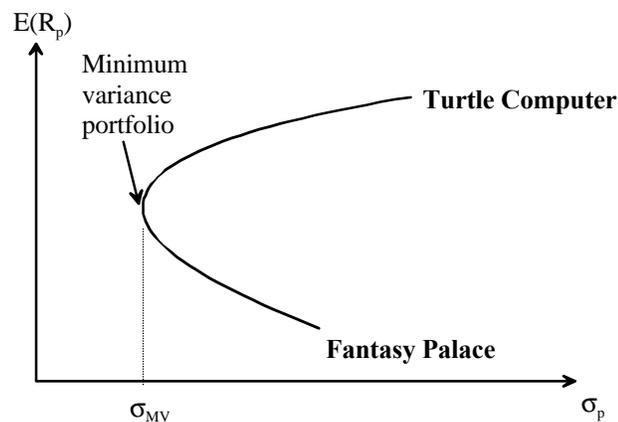
$$E(R_p) = (0.8 \times 0.15) + (0.2 \times 0.10) = 0.14$$

$$\sigma_p = \sqrt{(0.8^2 \times 0.4^2) + (0.2^2 \times 0.3^2) + (2 \times 0.8 \times 0.2 \times 0.4 \times 0.3 \times 0)} = 0.326$$

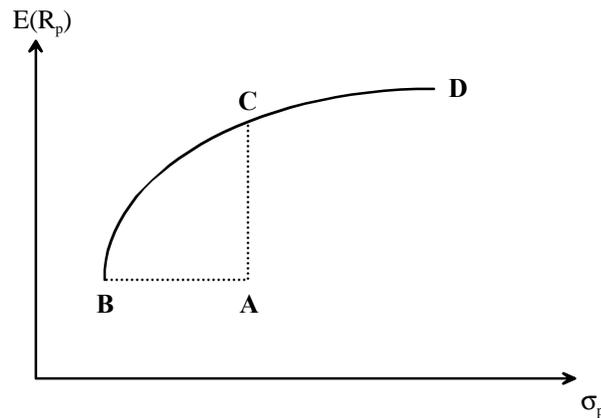
We can also calculate the expected returns and the standard deviations of the portfolio with various combinations of w_1 and w_2 using the same formulas. The following table presents the expected returns and standard deviations of the portfolio for some of these combinations:

w_1	w_2	Expected Return	Standard Deviation
1.0	0.0	15.0%	40.0%
0.8	0.2	14.0%	32.6%
0.5	0.5	12.5%	25.0%
0.2	0.8	11.0%	25.3%
0.0	1.0	10.0%	30.0%

When we plot the expected returns and standard deviations of the portfolio for all the various combinations of w_1 and w_2 , we will get the following the graph:



The curve depicted in the previous graph is known as the *efficient frontier*, which maps the risk and return relationship of a portfolio. It represents the different portfolios that are attainable by an investor. Portfolios that lie outside the efficient frontier (as represented by the area to upper left of the efficient frontier) are unattainable by the investor.



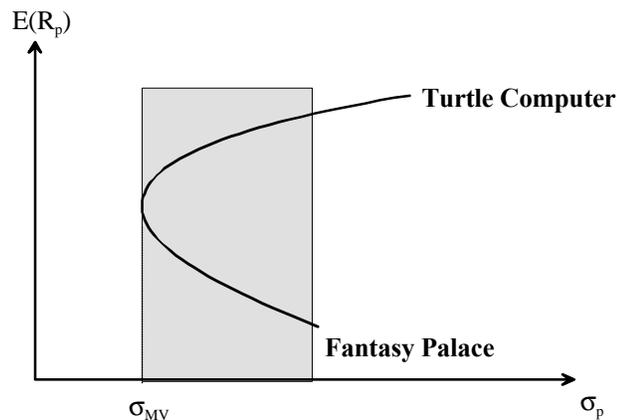
What can we say about the four different portfolios (A, B, C, and D) represented in the graph above?

1. **Portfolio A vs. portfolio B:** Portfolio B is a better choice because it offers the same return as portfolio A but at a lower level of risk.
2. **Portfolio A vs. portfolio C:** Portfolio C is a better choice because it offers a higher return for the same level of risk as portfolio A.
3. **Portfolio B vs. portfolio D:** It is difficult to choose between these two portfolios. From the graph we can see that portfolio D has a higher level of risk than portfolio B, but it also provides a higher return than portfolio D. The portfolio an investor picks will depend on his/her risk preference.

If you look at the efficient frontier closely, you will see that the risk level of the portfolio changes as its composition changes. The upper end of the frontier represents the case where the entire fund is invested in the riskier Turtle Computer. As the composition of the portfolio switches from only Turtle Computer to more and more Fantasy Palace, we will notice that the risk level of the portfolio falls because Fantasy Palace is a safer investment. However, the risk level of the portfolio “bottoms out” at a certain combination of Turtle Computer and Fantasy Palace. That particular combination that causes the risk level of the portfolio to “bottom out” (i.e. lowest level of risk) is known as

the *minimum variance portfolio*. It is not possible to lower the risk of the portfolio any further. If the composition of the portfolio continues to lean towards Fantasy Palace, the risk level of the portfolio will rise.

The interesting point about the graph is that it is possible for an investor to create a portfolio that has a risk level that is lower than the individual risk levels of the two stocks. This is depicted by the part of the efficient frontier that lies in the shaded area in the following graph:



How is it possible for an investor to create a portfolio that has a risk level lower than the individual risk levels of the two stocks it contains? This is the power of diversification, which allows an investor to lower the risk level of a portfolio beyond the individual risk levels of the stocks it contains. From the graphs above, we know the minimum variance portfolio is the one that achieves the maximum effect of diversification. The following formulas will help an investor determine the correct combination of w_1 and w_2 that will produce the minimum variance portfolio:

$$w_1 = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$$

$$w_2 = 1 - w_1$$

Using the information provided, we can easily determine the combination of Turtle Computer and Fantasy Palace that will produce the minimum variance portfolio (assuming they have a random relationship) as follows:

$$w_1 = \frac{0.3^2 - (0.3 \times 0.4 \times 0)}{0.4^2 + 0.3^2 - (2 \times 0.3 \times 0.4 \times 0)} = 0.36$$

$$w_2 = 1 - 0.36 = 0.64$$

Given the calculation, we know that an investor needs to invest 36% of his/her fund in Turtle Computer and 64% of the fund in Fantasy Palace if he/she wants to have a portfolio with the lowest possible level of risk.

Case 2: The correlation between Turtle Computer and Fantasy Palace is –1 and 1

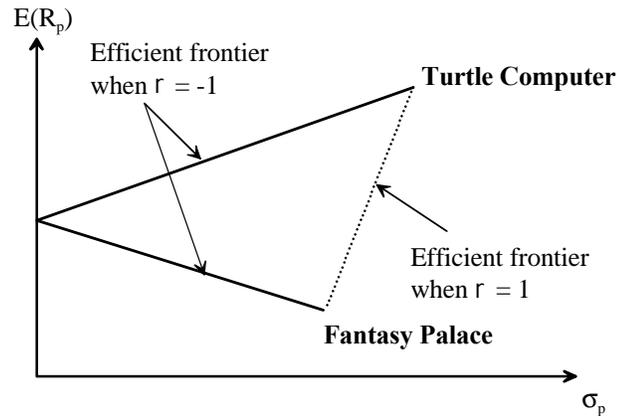
In the previous case, we have determined the expected returns and standard deviations of a portfolio that contains various combinations of Turtle Computer and Fantasy Palace (assuming that the correlation coefficient between the two stocks is zero). By graphing all the expected returns and standard deviations computed, we are able to depict the risk and return relationship of the portfolio with the efficient frontier.

In this particular case, we will redo the analysis by looking at two different scenarios: when the correlation coefficient between the two stocks is –1 and when the correlation coefficient is +1. It is important to note that changes in the correlation coefficient will have an impact on the risk level of the portfolio, but will have no impact on its expected return.

Using the same formulas as before, we can compute the expected returns and standard deviations of the portfolio (using different combinations of w_1 and w_2) for the two scenarios. The following table presents the results for some of these calculations:

w_1	w_2	$E(R_p)$	$\sigma_p (\rho = 1)$	$\sigma_p (\rho = -1)$
1.0	0.0	15.0%	40%	40%
0.8	0.2	14.0%	26%	38%
0.5	0.5	12.5%	5%	35%
0.2	0.8	11.0%	16%	32%
0.0	1.0	10.0%	30%	30%

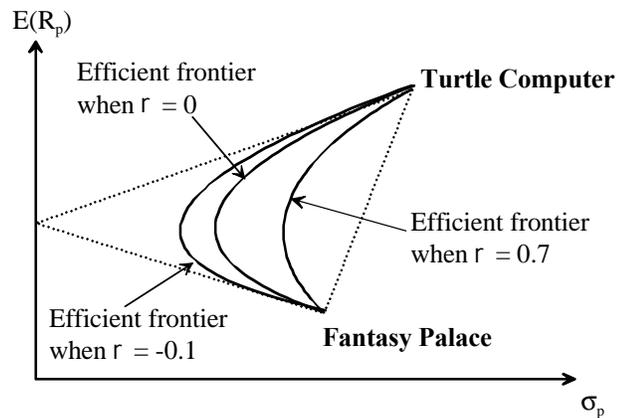
We can also obtain the efficient frontiers for these two particular scenarios by plotting the expected returns and standard deviations computed for various combinations w_1 and w_2 . The following graph depicts the two efficient frontiers when the correlation coefficients are -1 and 1 :



The graph above confirms some of our earlier discussions: diversification is most effective when the correlation coefficient is -1 and it is least effective (or not effective at all) when the correlation coefficient is $+1$. In the case of a perfect negative correlation coefficient (i.e. $\rho = -1$), it is possible for an investor to create a portfolio that has no risk at all. On the other hand, in the case of a perfect positive correlation coefficient (i.e. $\rho = +1$), the only way for the investor to minimize risk is to invest the entire fund in the less risky stock (i.e. Fantasy Palace).

Relationships among the efficient frontiers associated with different correlation coefficients

So far, we have looked at 3 different efficient frontiers of a portfolio under the assumption that the correlation coefficient between the two stocks in the portfolio had a value of -1 , 0 , and $+1$. From our earlier discussion, we know that a correlation coefficient has to take on a value between -1 and $+1$. As depicted in the graph in Case 2, the efficient frontiers associated with correlation coefficients of -1 and $+1$ will form a “triangle”. This “triangle” represents the boundary that contains all the efficient frontiers associated with other values of correlation coefficient. The following graph depicts how the efficient frontiers associated with correlation coefficients of -0.1 , 0 , and 0.7 relate to the “triangular” boundary:

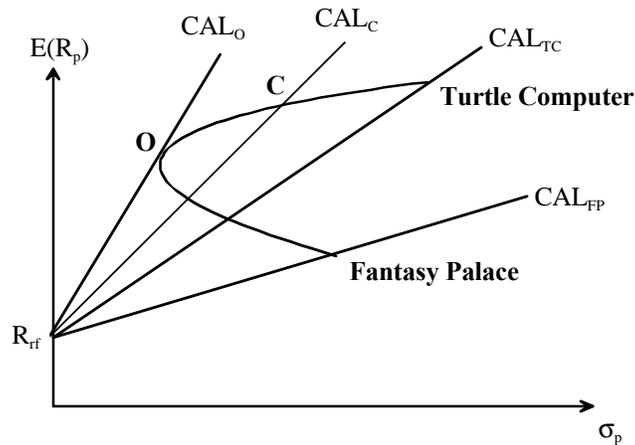


It is important to remember that the shapes of the efficient frontiers depicted in this section are based on a portfolio that is invested in Turtle Computer and Fantasy Palace. The shapes of the efficient frontiers will be different for other portfolios that have invested in other risky assets that have different expected returns and standard deviations.

A Portfolio Containing Multiple Risky Assets and a Risk-free Asset

In the last two sections, we looked at two different portfolios: one containing a risky asset and a risk-free asset, and the other containing multiple risky assets. We have established the risk and return relationship (through the Capital Allocation Line) with the first portfolio, and the concept of diversification (through the efficient frontier) with the second portfolio. In this section, we will look at a portfolio that contains multiple risky assets and a risk-free asset.

To simplify our analysis, we will look at a situation where there are only two risky assets (i.e. Turtle Computer and Fantasy Palace) and a risk-free asset available. Based on our earlier analysis, we know that if an investor decides to invest only in Fantasy Palace and the risk-free asset, the risk and return relationship of the portfolio will be represented by the capital allocation line denoted CAL_{FP} in the following graph. From our earlier discussion, we also know the investor can determine CAL_{FP} by simply connecting the point representing the risk-free asset and the point representing Fantasy Palace. The investor will be somewhere along CAL_{FP} depending on how he/she allocates the funds between the two assets. Similarly, if the investor decides to invest only in the risk-free asset and Turtle Computer, the risk and return relationship of the portfolio will be represented by the capital allocation line denoted CAL_{TC} .



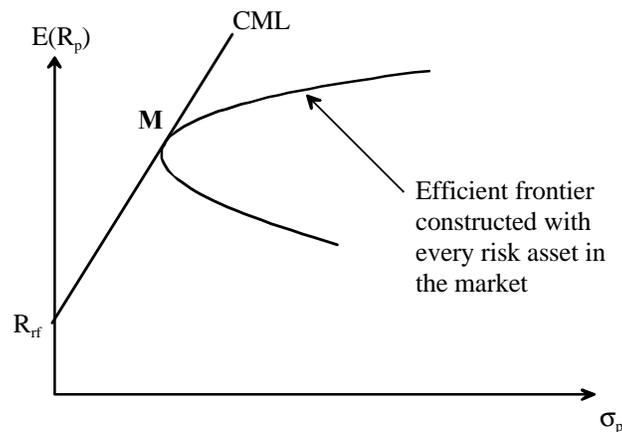
Based on the graph above, we know that an investor will always pick the portfolio that contains Turtle Computer and the risk-free asset over the portfolio that contains Fantasy Palace and the risk-free asset. This is because CAL_{TC} is above CAL_{FP} , which means that for the same level of risk, the investor will receive a higher return from the portfolio that contains Turtle Computer and the risk-free asset compare to the return from the portfolio that contains Fantasy Palace and the risk-free asset.

Using the same logic, we know that the investor will be even better off if he/she decides to invest in the risk-free asset and a certain combination of Turtle Computer and Fantasy Palace (represented by portfolio C), because its capital allocation line (CAL_C) is above that of CAL_{TC} . We know the investor can keep on improving his/her situation by changing the combination of Turtle Computer and Fantasy Palace, which shifts the capital allocation line of the portfolio leftward.

In this particular situation, the investor will reach the “best situation” when he/she invests in the risk-free asset and a certain combination of Turtle Computer and Fantasy Palace (represented by portfolio O). The capital allocation line of this particular investment is represented by CAL_O , which is tangent to the efficient frontier. The investor cannot shift the capital allocation line further to the left because that will involve investing in unattainable portfolios. Hence, portfolio O is also known as the ***optimal risky portfolio***. In other words, this is the portfolio that provides the best risk and return tradeoff for the investor.

At this point in time, it is important to point out the difference between the minimum variance portfolio and the optimal risky portfolio (both located on the efficient frontier). The objective of an investor interested in the minimum variance portfolio is to minimize the risk of the portfolio. Whereas the objective of an investor interested in the optimal risky portfolio is to obtain the best risk and return tradeoff available.

We will now expand our analysis by looking at an environment that contains all the available risky assets in the market. Using the same procedure, we can map out the efficient frontier for this particular scenario. Similarly, it would be best for the investor to be located on the capital allocation line that is tangent to the efficient frontier. This particular capital allocation line is also known as the **Capital Market Line (CML)**, and the optimal risky portfolio associated with it is known as the **market portfolio** (i.e. a portfolio with all the risky assets in proportion to their market values). The capital market line and the market portfolio (represented by portfolio M) are depicted in the graph below:



The formula for the capital market line (CML) is very similar to that of the capital allocation line (CAL):

$$E(R_p) = R_{rf} + \left[\frac{E(R_M) - R_{rf}}{\sigma_M} \right] \sigma_p$$

It represents the risk and return relationship of a portfolio that contains multiple risky assets and the risk-free asset. Once again, we can look at the return of the portfolio by breaking it down into two components:

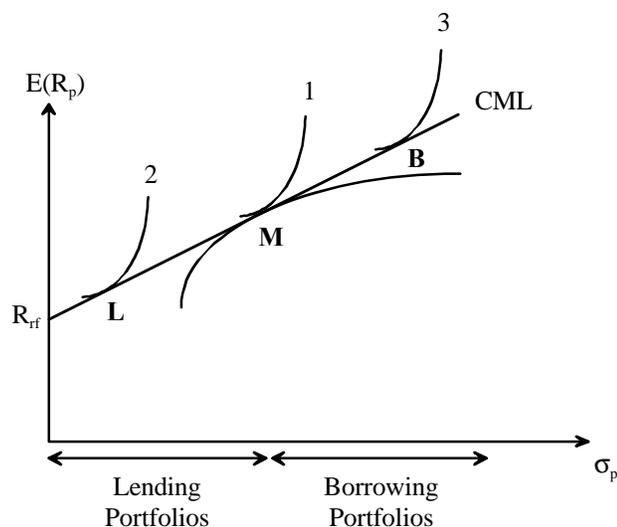
1. The first component is the risk-free rate, which represents the minimum compensation an investor is expected to receive.
2. The second component is the investor's compensation for taking the risk. This can be further broken down into two items:

(a) $\frac{E(R_M) - R_{rf}}{\sigma_M}$ = compensation for each unit of risk taken \Rightarrow market price of risk

(b) σ_p = amount of risk in the portfolio

The Capital Market Line and an Investor's Investment Decisions

Based on our earlier discussion, the capital market line (CML) provides the best risk and return tradeoff for an investor. Along the CML, a potential portfolio for the investor will be the market portfolio. However, it is not necessary for the investor to pick that portfolio because he/she can pick any portfolio along the CML. This investment decision will be heavily influenced by the investor's risk preference. The following graph illustrates how the risk preference of an investor affects his/her investment decisions:



1. If the investor's risk preference is represented by indifference curve 1, he/she will pick portfolio M, which is the market portfolio. In other words, the investor's

portfolio would be made up of all the risky assets in proportion to their market values, but it would not contain any risk-free asset.

2. If the investor's risk preference is represented by indifference curve 2, he/she will pick portfolio L, which has a lower level of risk than the market portfolio. The only way to reduce the risk of the portfolio is to include the risk-free security in it. In essence, the investor is "lending" part of his/her money at the risk-free rate. As a result, portfolio L is also known as a lending portfolio. From the graph above, we know that any portfolio along the CML that is below the market portfolio can be termed a lending portfolio.
3. If the investor's risk preference is represented by indifference curve 3, he/she will pick portfolio B, which has a higher level of risk than the market portfolio. The only way to increase the risk of the portfolio is to borrow at the risk-free rate to increase the investment base (i.e. margin trading). As a result, portfolio B is also known as a borrowing portfolio. From the graph, we know that any portfolio along the CML that is above the market portfolio can be termed a borrowing portfolio.

According to James Tobin's *Separation Theorem*, an investor's decision making process is actually made up of two separate decisions:

- i. To be on the CML, the investor initially decides to invest in the market portfolio (i.e. portfolio M). This is the investment decision.
- ii. Based on the investor's risk preference, he/she makes a separate financing decision on whether to lend or borrow at the risk-free rate to get to the desired point.

In the above three scenarios, we have seen how an investor can adjust his/her return or risk by borrowing or lending with a risk-free asset against a single optimal risky portfolio (i.e. the market portfolio). From the graph, we know that in the situation where a risk-free asset is not available, the investor can only reduce the risk of the portfolio by shifting his/her funds towards "safer" risky assets, which will move the investor downward along the efficient frontier. However, when the investor has access to the risk-free asset, he/she can reduce the risk of the portfolio by investing or lending at the risk-free rate, which will move him/her downward along the CML. We know both strategies will reduce the risk of

the portfolio at the expense of a lower return. If you look at the graph carefully, you will realize that the portfolios on the CML provide a higher return than the portfolios on the efficient frontier for the same level of risk. In other words, the availability of the risk-free asset helps an investor reduce the risk of his/her portfolio but also helps preserve most of the return of the portfolio.

Similarly, we know that in the absence of the risk-free asset, the investor can improve the return of the portfolio by shifting his/her funds towards “riskier” risky assets, which will move the investor upward along the efficient frontier. However, with the presence of the risk-free asset, the investor can improve the return of the portfolio by borrowing at the risk-free rate, which will move the investor along the CML. We know both strategies will improve the return of the portfolio at the expense of a higher level of risk. If you look at the graph carefully, you will realize that the portfolios on the CML experienced a smaller increase in risk level than the portfolios on the efficient frontier for the same level of increase in return. In other words, the availability of the risk-free asset helps an investor improve the return of his/her portfolio without taking on as much risk.

Conclusion

In this chapter, we have looked at three different portfolios and we learned three different lessons from these portfolios:

1. With a portfolio that contains a risky asset and a risk-free asset, we derived the basic relationship between risk and return through the capital allocation line. We found that the return of a portfolio can be broken down into two components: a guaranteed return and a compensation for taking risk.
2. With a portfolio that contains multiple risky assets, we derived the efficient frontier that helped explain the concept of diversification. In other words, it is possible for an investor to put together a portfolio that has a risk level lower than the individual risk levels of the assets it contained.
3. With a portfolio that contains multiple risky assets and a risk-free asset, we showed that it is possible for an investor to improve his/her risk or return situation by lending or borrowing at the risk-free rate.

The lessons we have learned in the last portfolio will provide the foundation that will help us understand the Capital Asset Pricing Model in the next chapter (CAPM). The CAPM, despite its flaws, is one of the most commonly used tools in the securities industry for pricing financial assets.

Appendix 5. Using Scientific and Financial Calculators to Determine Correlation Coefficient

The easiest way to learn how to use the calculators is to illustrate the procedure with an example. Suppose you are given the following 10 pairs of observations:

X	12	20	23	17	53	27	42	10	28	11
Y	20	17	54	13	19	47	62	43	18	28

I. Scientific Calculator

Almost all the Casio scientific calculators will allow you to calculate simple statistical values (such as average and standard deviation), but only some of them will allow you to perform simple linear regression analysis. Make sure you buy a Casio scientific calculator with as many functions as possible because you need one that will perform simple linear regression analysis. The cover of the box will tell you how many functions the calculator performs.

1. Change the mode of function of the calculator to LR. You do this by pressing the **MODE** key and another key. Your calculator should tell you what that key is. Once you have done that, you should see the symbol **LR** on the screen.
2. You should always clear the calculator's memory before inputting any number. You do so by pressing the **SHIFT** key follow by pressing the **AC** key.
3. Once you have changed the mode of operation and clear the memory, you are ready to input the data. You have to enter the numbers in pairs, and you always enter the x value of the pair first and the y value second. You enter each pair of number with the following procedure:
 - a. Enter the x value and press the **x_D,y_D** key.
 - b. Enter the y value and press the **DATA** key.
4. After you have entered all the numbers, you can find the sample correlation coefficient by pressing the **SHIFT** key follow by pressing the **r** key. The correlation coefficient of the 10 pairs of numbers in our example is 0.1629.

II. Financial Calculator

The procedure for calculating the sample correlation coefficient is very similar to the procedure for calculating the average and the sample standard deviation. The only difference is the way the numbers are being entered.

1. You need to first get the calculator ready to input the data. You do so by pressing the **2nd** key follow by pressing the **Data** key. You should see the symbol **X01** on the screen.
2. Before you start inputting the numbers, you need to clear the calculator's memory by pressing the **2nd** key follow by pressing the **CLR Work** key.
3. Since this is a series of pairs of numbers, you need to enter each pair of numbers as x value and y value. To enter the first pair, you put in the first number for x and press the **ENTER** key. Press the \downarrow key and you will see the symbol **Y01** on the screen. Put in the first number for y and press the **ENTER** key. Press the \downarrow key and you will see the symbol **X02** on the screen. Put in the second number for x and press the **ENTER** key follow by the second number for y and press the **ENTER** key. Press the \downarrow key and you will see **X03** on the screen. Continue to enter the rest of the pairs of numbers using the same procedure.
4. Once you have entered all the pairs of numbers, you can find the correlation by first pressing the **2nd** key follow by pressing the **Stat** key. You will see the symbol **LIN** on the screen. Keep pressing the \downarrow key until you get the symbol **r** on the screen for the correlation coefficient. You should see 0.1629 if you have entered the pairs of numbers in our example